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THE HAMILTON-JACOBI EQUATION IN THE NEIGHBOURHOOD OF A POSITION OF EQUILIBRIUM*

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The Hamilton-Jacobi equation is considered in the region of a position of equilibrium, which is not a local minimum of the potential energy. If the potential has a local maximum, the Hamilton-Jacobi equation has a continuous or analytical solution in the neighbourhood of the position of equilibrium /1, 2/. In the case of a saddle point a solution is sought in complex form. The well-known Bol theorem /3/ on the asymptotic motions of a natural mechanical system in the neighbourhood of a position of equilibrium is obtained as a corollary. The problem of the presence of continuous solutions of the Hamilton-Jacobi equation in cases of degeneracy is investigated.

1. The complex solutions of the Hamilton-Jacobi equation. We will consider a holonomic mechanical system with n degrees of freedom (x are its generalized coordinates, and y are its generalized momenta), the motion of which is described by the following canonical equations:

$$x^{\cdot} = \partial H/\partial y,\ y^{\cdot} = -\partial H/\partial x;\ x = (x_1, \ldots, x_n),\ y = (y_1, \ldots, y_n) \tag{1.1}$$

where $H(x, y)$ is the Hamilton function. Then the corresonding abbreviated Hamilton-Jacobi equation has the form

$$H(x, \partial S/\partial x) = h \tag{1.2}$$

where h is an arbitary constant (the energy constant).

Suppose the function $S = S_1 + iS_2;\ S_1, S_2: R^n\{x\} \to R,\ i = \sqrt{-1}$ is obtained which satisfies Eq.(1.2). We introduce the manifold $K = \{y = \partial S_1/\partial x,\ \partial S_2/\partial x = 0\}$ in phase space $R^{2n}\{x, y\}$.

Theorem 1. If the point $(x^\circ, y^\circ) \in K$, the solution of the canonical system with initial condition (x°, y°) as a whole lies on K.

Proof. Note that in view of Eqs.(1.1)

$$\frac{d}{dt}\frac{\partial S_2}{\partial x_i} = \Sigma_2,\quad \frac{d}{dt}\left(\frac{\partial S_1}{\partial x_i} - y_i\right) = \Sigma_1 + \frac{\partial H}{\partial x_i}\quad \Sigma_\alpha = \sum_{j=1}^n \frac{\partial^2 S_\alpha}{\partial x_j \partial x_i}\frac{\partial H}{\partial y_j},$$
$$\alpha = 1, 2;$$

By differentiating (1.2) with respect to x_i, we obtain $\Sigma_2|_K = 0,\ (\Sigma_1 + \partial H/\partial x_i)_K = 0$, which proves the theorem.

Note that if $S_2 \equiv 0$, the manifold K is Lagrangian.

2. Solutions in the neighbourhood of non-degenerate positions of equilibrium of natural systems and the manifolds of the asymptotic motions. Consider a natural mechanical system with analytical Hamiltonian

$$H = \frac{1}{2}\sum_{i,j=1}^n a^{ij}(x)\, y_i y_j + \Pi(x)$$

where (a^{ij}) is a positive-definite matrix of the kinetic energy, while $\Pi(x)$ is the potential energy of the system. The Hamilton-Jacobi equation at the zero energy level has the form

$$\frac{1}{2}\sum_{i,j=1}^n a^{ij}(x)\frac{\partial S}{\partial x_i}\frac{\partial S}{\partial x_j} + \Pi(x) = 0 \tag{2.1}$$

We will assume that $x = 0$ is the position of equilibrium ($d\Pi(0) = 0$) and $\Pi(0) = 0$. Without loss of generality, we will assume that in the neighbourhood of a position of equilibrium

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$$\Pi = \frac{1}{2}\sum_{i=1}^{n}\lambda_i x_i^2 + \sum_{k=3}^{\infty}\Pi_k, \quad a^{ij} = \delta^{ij} + \sum_{k=1}^{\infty} A_k^{ij}$$
$$\Pi_k = \sum_{\alpha_1+\ldots+\alpha_n=k} p_{\alpha_1\ldots\alpha_n} x_1^{\alpha_1}\ldots x_n^{\alpha_n}$$
$$A_k^{ij} = \sum_{\alpha_1+\ldots+\alpha_n=k} a^{ij}_{\alpha_1\ldots\alpha_n} x_1^{\alpha_1}\ldots x_n^{\alpha_n}$$

(Π_k, A_k^{ij} are forms of degree k, while $\|\delta^{ij}\|$ is the unit matrix).

Eq.(2.1) was investigated in /1/ when the potential energy in the position of equilibrium has a non-degenerate maximum. We will consider the general case when the potential energy in the position of equilibrium does not necessarily have a maximum.

The solution of Eq.(2.1) will be sought in the form of a series

$$S = \frac{1}{2}\sum_{i=1}^{n}\Lambda_i x_i^2 + \sum_{k=3}^{\infty} S_k, \quad S_k = \sum_{\alpha_1+\ldots+\alpha_n=k} a_{\alpha_1\ldots\alpha_n} x_1^{\alpha_1}\ldots x_n^{\alpha_n} \tag{2.2}$$

Substituting it into Eq.(2.1) we obtain $\Lambda_i = \pm\sqrt{-\lambda_i}$ $(i = 1, \ldots, n)$. We will take $\Lambda_i = -\sqrt{-\lambda_i}$. Then the coefficients of the form S_k $(k = 3, 4, \ldots)$ are determined from the recurrent relation

$$\sum_{\alpha_1+\ldots+\alpha_n=k}\left(\sum_{i=1}^{n}\Lambda_i\alpha_i\right) a_{\alpha_1\ldots\alpha_n} x_1^{\alpha_1}\ldots x_n^{\alpha_n} = -\Pi_k - \frac{1}{2}\sum_{i=1}^{n}\sum_{l=3}^{k-1}\frac{\partial S_l}{\partial x_i}\times$$
$$\frac{\partial S_{k+2-l}}{\partial x_i} - \frac{1}{2}\sum_{i,j=1}^{n}\sum_{m=1}^{k-2} A_m^{ij}\sum_{l=2}^{k-m}\frac{\partial S_l}{\partial x_i}\frac{\partial S_{k+2-m-l}}{\partial x_j}$$

Since Λ_i are either real or imaginary numbers, then, if $\lambda_i \neq 0$, $i = 1, \ldots, n$ (the position of equilibrium is non-degenerate), the expression $\alpha_1\Lambda_1 + \ldots + \alpha_n\Lambda_n$ can never vanish for negative integer α_i, such that $\alpha_1 + \ldots + \alpha_n = k$ $(k = 3, 4, \ldots)$. Consequently, the forms S_k are uniquely defined. The convergence of the power series obtained can be proved by the method employed in /1/.

Thus the following theorem holds.

Theorem 2. In the neighbourhood of a non-degenerate position of equilibrium a solution of Eq.(2.1) exists, which can be represented in the form of the converging series (2.2).

Note that if $S(x)$ is a solution, then $-S(x)$ is also a solution. Suppose $\lambda_i < 0$ for $i = 1, \ldots, m$ and $\lambda_i > 0$ for $i = m+1, \ldots, n$ $(0 \leqslant m \leqslant n)$. We then have a solution of the form $S = S_1 + iS_2$, where

$$S_1 = -\frac{1}{2}\sum_{i=1}^{m}\sqrt{-\lambda_i}x_i^2 + \bar{S}_1(x), \quad d^2\bar{S}_1(0) = 0$$
$$S_2 = -\frac{1}{2}\sum_{i=m+1}^{n}\sqrt{\lambda_i}x_i^2 + \bar{S}_2(x), \quad d^2\bar{S}_2(0) = 0$$

Theorem 3. a) If $m = 0$ ($x = 0$ is the point of minimum potential energy), the invariant manifold from Theorem 1 is degenerate in the state of equilibrium.

c) If $m > 0$, then the m-dimensional manifolds

$$K_\pm^m = \{y = \pm\partial S_1/\partial x,\ 0 = \partial S_2/\partial z\},\ z = (x_{m+1}, \ldots, x_n)$$

are continuously filled by trajectories which asymptotically approach the state of equilibrium as $t \to \pm\infty$.

Proof. Case a) is trivial. The case when $m = n$ is considered in /1/ (systems with a continuous potential were considered previously in /2/). Suppose $0 < m < n$. It follows from (2.1) that the real and imaginary parts of the solution satisfy the relation

$$\sum_{i,j=1}^{n} a^{ij}\frac{\partial S_1}{\partial x_i}\frac{\partial S_2}{\partial x_j} = 0 \tag{2.3}$$

Consider the system of equations

$$\frac{\partial S_2}{\partial x_j} = -\sqrt{\lambda_i} x_j + \frac{\partial \bar{S}_2}{\partial x_j} = 0, \quad j = m+1, \ldots, n \tag{2.4}$$

Since the Jacobian

$$\left.\frac{D(\partial S_2/\partial x_{m+1}, \ldots, \partial S_2/\partial x_n)}{D(x_{m+1}, \ldots, x_n)}\right|_0 = (-1)^{n-m}\sqrt{\lambda_{m+1}}\ldots\sqrt{\lambda_n} \neq 0$$

the system of Eqs.(2.4) can be solved in the neighbourhood of the origin of coordinates. As a result we obtain the functions

$$x_{m+l} = f_l(x_1, \ldots, x_m) \ (f_l(0) = 0), \ l = 1, \ldots, n-m$$

We will differentiate the function. $S_{2*} = S_2(x_1, \ldots, x_m, f_1(x_1, \ldots, x_m), \ldots, f_{n-m}(x_1, \ldots, x_m))$ in view of the system

$$y_i = \left(\frac{\partial S_1}{\partial x_i}\right)_* = -\sqrt{-\lambda_i} x_i + \left(\frac{\partial \bar{S}_1}{\partial x_i}\right)_*, \quad i = 1, \ldots, m \tag{2.5}$$

We obtain

$$\frac{dS_{2*}}{dt} = \sum_{i=1}^{n}\sum_{j=1}^{m} a_*^{ij}\left(\frac{\partial S_1}{\partial x_i}\right)_*\left(\frac{\partial S_2}{\partial x_j}\right)_*$$

which, in view of relation (2.3), is identically equal to zero. Since $\sqrt{-\lambda_1}\alpha_1 + \ldots + \sqrt{-\lambda_m}\alpha_m \neq 0$ for integer negative values of α_i such that $\alpha_1 + \ldots + \alpha_m = k$ $(k = 1, 2, \ldots)$, then $S_{2*} \equiv \mathrm{const} = 0$.

Consequently,

$$K_\pm^m = \{y = \pm\partial S_1/\partial x, \ \ x_{m+l} = f_l(x_1, \ \ldots, \ x_m), \ \ l = 1, \ \ldots, \ n-m\}$$

are invariant m-dimensional manifolds from Theorem 1. Since $\pm S_{1*}$ has a maximum (a minimum) and the derivative with respect to time, in view of system (2.5), in the neighbourhood of the position of equilibrium is positive, $K_\pm^m$ consists of phase trajectories that asymptotically approach a state of equilibrium as $t \to \pm\infty$.

Kneser /4/ was the first to prove the existence of asymptotic motions of systems with two degrees of freedom in the neighbourhood of non-degenerate positions of equilibrium, in which the potential energy has a local maximum. Asymptotic motions in the general non-degenerate case have been investigated by Bol /3/. Theorem 3 connects the manifolds of the phase trajectories of the asymptotic motions with the solutions of the Hamilton-Jacobi equation.

We will now consider the more general case of seminatural systems instead of natural systems.

Suppose the Hamilton-Jacobi equation has the form

$$\frac{1}{2}\sum_{i,j=1}^{n} a^{ij}\left(\frac{\partial S}{\partial x_i} - b_i\right)\left(\frac{\partial S}{\partial x_j} - b_j\right) + \Pi(x) = 0$$

where $db_i(0) = 0$, $i = 1, \ldots, n$, i.e. the linearized system is gyroscopically unconnected. Theorem 2 remains true: in the general non-degenerate case the solutions have the form $S^\pm = S_1^\pm + iS_2^\pm$ $(S^- \neq -S^+)$, where

$$S_1^\pm = -\frac{1}{2}\sum_{i=1}^{m} \pm\sqrt{-\lambda_i} x_i^2 + \bar{S}_1^\pm(x),$$

$$S_2^\pm = -\frac{1}{2}\sum_{i=m+1}^{n} \pm\sqrt{\lambda_i} x_i^2 + \bar{S}_2^\pm(x)$$

As in Theorem 3, they determine the invariant manifolds of asymptotically phase trajectories as $t \to +\infty$ and $t \to -\infty$. Unlike the natural case, the phase trajectories are projected onto the non-coincident trajectories of asymptotic motions as $t \to \pm\infty$.

3. The degenerate case. In the general degenerate case, Eq.(2.1) has no analytical solution in the neighbourhood of the point $x = 0$. Non-analytical solutions can exist (see, for example, /1/). We will consider the fairly general degenerate case when the expansion of the potential energy in the neighbourhood of a position of equilibrium begins with a form of even order of the form $2\Pi_{2m} = -a^2|x|^{2m}$, $m > 2$, $a^2 = \mathrm{const}$. For simplicity we will assume that $n = 2$. Changing to polar coordinates $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$ we can convert Eq.(2.1) to the form

$$(1 + A^1)\left(\frac{\partial S}{\partial r}\right)^2 + \frac{2}{r}A^2\frac{\partial S}{\partial r}\frac{\partial S}{\partial \varphi} + \frac{1}{r^2}(1 + A^3)\left(\frac{\partial S}{\partial \varphi}\right)^2 = \tag{3.1}$$

$$a^2r^{2m} + r^{2m+1}\sum_{j=1}^{\infty} r^{j-1}\Pi_{2m+j}(\varphi)$$

$$A^l = \sum_{k=1}^{\infty} r^k A^l_{2+k}(\cos\varphi, \sin\varphi), \qquad l = 1, 2, 3$$

where the polynomials $A^l_{2+k}(\cos\varphi, \sin\varphi)$ depend to a certain extent on the coefficients A_k^{ij} of the form of the kinetic energy.

The simplified equation

$$\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \varphi}\right)^2 = a^2r^{2m}$$

has the solution $S = ar^{m+1}/m + 1$. The solution of the complete equation will be sought in the form

$$S = \frac{a}{m+1}r^{m+1} + \sum_{j=1}^{\infty} r^{m+1+j}S_{m+1+j}(\varphi) \tag{3.2}$$

Theorem 4. A solution of Eq.(3.1) exists, which can be represented in the form of the series (3.2), which converges when r is fairly small.

The following lemma holds.

Lemma. If a solution of the form (3.2) of Eq.(3.1) exists, where

$$A^l = A^l(\varepsilon, r, \varphi), \; l = 1, 2, 3 \tag{3.3}$$

for fairly small ε, which converges when $r \leqslant r_0$, then the corresponding solution of Eq.(3.1) converges when $r \leqslant \varepsilon r_0$.

Proof of the theorem. In view of the lemma it is sufficient to prove that a solution of Eq.(3.1), (3.3) exists. Suppose A is a space of functions which can be represented when $r \leqslant r_0$ by the absolutely converging series.

$$f = \sum_{j=0}^{\infty} r^{m+1+j}\Phi_{m+1+j}(\varphi); \qquad \Phi_{m+1+j} \in C^\infty[0, 2\pi]$$

are infinitely differentiable 2π-periodic functions. In this space we specify the norm

$$\|f\|_1 = \sup_k\{\|f\|_1^{(k)}\}, \quad \|f\|_1^{(k)} = \sup_\varphi \sum_{j=0}^{\infty}(m + 1 + j)r_0^{m+1+j}|\Phi^{(k)}_{m+1+j}|$$

where $\Phi^{(k)}$ is a derivative of the k-th order. We will similarly introduce a space B

$$f = \sum_{j=0}^{\infty} r^{2m+j}\Phi_{2m+j}(\varphi), \qquad \|f\|_2 = \sup_k\{\|f\|_2^{(k)}\}$$

$$\|f\|_2^{(k)} = \sup_\varphi \sum_{j=0}^{\infty} r_0^{2m+j}|\Phi^{(k)}_{2m+j}|$$

The spaces A and B are Banach spaces.

We write Eqs.(3.1), (3.3) in the form $F(S, \varepsilon) = 0$ and we will consider F as the mapping of a fairly small neighbourhood

$$V = \{(S, \varepsilon): \|S - S_0\|_1 < \delta, \; |\varepsilon| < \varepsilon_0\}, \; S_0 = ar^{m+1}/m + 1$$

into B.

The following assertions hold: 1) $F(S_0, 0) = 0$. 2) The mapping F is continuous at the point $(S_0, 0)$. 3) The derivative $F_S'(S, \varepsilon)$ exists in V and is continuous at the point $(S_0, 0)$.

We will show that the linear operator $F_S'(S_0, 0) = ar^m\partial/\partial r$ has a limited inverse. In fact, suppose

$$v = \sum_{j=0}^{\infty} r^{2m+j}V_{2m+j} \in B$$

Then the equation $F_S'(S_0, 0)\; u = v$ has the solution

$$u = \frac{1}{a}\sum_{j=0}^{\infty}\frac{r^{m+1+j}}{m+1+j}V_{2m+j}, \quad \|u\|_1 \leqslant c\|v\|_2, \quad c = \frac{1}{ar_0^{m-1}}$$

Consequently, in view of the theorem on an implicit function /5/, for small ε in the space A there is a solution $S(r, \varphi, \varepsilon)$ of the equation $F(S(r, \varphi, \varepsilon), \varepsilon) = 0$, which differs only slightly from S_0. The theorem is proved.

Returning to the old variables x_1 and x_2, we obtain at least a function of the class C^m. The solutions obtained, as in Sect.2, define the manifolds $y = \pm\partial S/\partial x$ of the phase trajectories of asymptotic motions (compare with /2/).

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PERIODIC SOLUTIONS OF HAMILTONIAN SYSTEMS IN CERTAIN DEGENERATE CASES*

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Periodic solutions of a canonical system of differential equations with a special type Hamiltonian, i.e. of the so-called fundamental problem of dynamics /1/, are investigated. A method of constructing the conditions of periodicity of the solutions is given and a non-linear analysis of the solutions is carried out. The method enables the Poincare's classical conditions of existence, as well as of the new conditions of existence of periodic solutions in degenerate cases to be derived. The cases of degeneracy discussed here appear very frequently in various problems of dynamics. The results obtained are illustrated by finding new periodic solutions for the problem of the motion of a heavy rigid body about a fixed point.

1. Formulation of the problem.

Consider the following system of canonical differential equations:

$$\frac{d\mathbf{I}}{dt} = \frac{\partial H}{\partial \boldsymbol{\varphi}^T}, \quad \frac{d\mathbf{J}}{dt} = \frac{\partial H}{\partial \boldsymbol{\psi}^T}, \quad \frac{d\boldsymbol{\varphi}}{dt} = -\frac{\partial H}{\partial \mathbf{I}^T}, \quad \frac{d\boldsymbol{\psi}}{dt} = -\frac{\partial H}{\partial \mathbf{J}^T} \tag{1.1}$$

$$\mathbf{I} = (p_1, \ldots, p_l)^T, \quad \mathbf{J} = (p_{l+1}, \ldots, p_N)^T, \quad \mathbf{p} = (p_1, \ldots, p_N)^T = (\mathbf{I}, \mathbf{J})^T$$

$$\boldsymbol{\varphi} = (q_1, \ldots, q_l)^T, \quad \boldsymbol{\psi} = (q_{l+1}, \ldots, q_N)^T,$$

$$\mathbf{q} = (q_1, \ldots, q_N)^T = (\boldsymbol{\varphi}, \boldsymbol{\psi})^T$$

$$H(\mathbf{p}, \mathbf{q}, t, \mu) \equiv H_0(\mathbf{I}) + \mu H_1(\mathbf{p}, \mathbf{q}, t) + \ldots, \ |\mu| \ll 1 \tag{1.2}$$

Let H be an analytic function of the position variables $\mathbf{p}$, the canonical angle variables $\mathbf{q}$ and the time t, in the region $D \times T^N \times T^1$, where D is a bounded connected region $R^N\{p_1, \ldots, p_N\}$ of the N-dimensional plane, $T^N\{q_1, \ldots, q_N \bmod 2\pi\}$ is an N-dimensional torus and $T^1\{t \bmod T_0\}$. Then the functions $H_i(\mathbf{p}, \mathbf{q}, t)$ $(i \geqslant 1)$ can be expanded in convergent Fourier series over the multiple angle variables q and Ωt ($\Omega = 2\pi/T_0$ is the fundamental frequency and T_0 is the period)

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